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BIOGRAPHY.

HUDSON A. WOOD.

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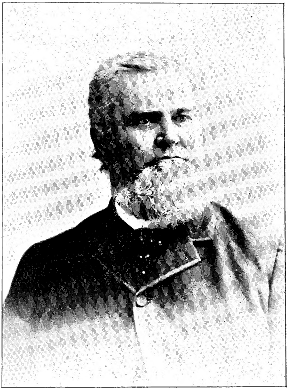
HUDSON A. WOOD, now Professor of Mathematics in the *Stevens School*, Hoboken, New Jersey, was born near Smyrna, New York, May 10, 1841. He is the middle one of a family of nine children; his brother, Professor DeVolson Wood, whose biography appeared in the September-October (1895) number of the MONTHLY, is the eldest. He was brought up on the farm, and early knew what hard work meant. He attributes his robust constitution to the vigorous exercise of his younger days. In the district school near his home, which he attended during the winter months, he acquired his early education. He evinced an unspeakable desire for study; and many a time, after a day's hard work, did he drop to sleep while poring over some book. At the age of fifteen, he spent his first winter away at school. When seventeen, he taught the district school adjoining his home; and, at the same time, he was initiated into the mysteries and pleasantries of boarding around. At the age of twenty he had taught a district school, a village school, and had completed the studies prescribed for the Freshman Class in Madison (now Colgate) University, at Hamilton, New York.

The year 1861, when Mr. Wood was twenty years of age, marks the beginning of the Civil War. A Company was raised at Hamilton, composed in part of students of the University. In this Company, afterward one of the Companies of the 61st Regiment of New York Volunteers, Mr. Wood enlisted. He was in the service nearly two years, and was engaged in six battles. His regiment took



HUDSON A. WOOD.

an active part in "The Seven-Days Battles" around Richmond, and sustained heavy losses. In the battle of Frazer's Farm, Mr. Wood had a ball shot through his coat, another through his haversack, and also received two slight flesh-wounds. More than one-half of the regiment fell in this desperate encounter at night-fall. He assisted in saving the colors of the regiment, for which he was promoted. At the battle of Fair Oaks, Mr. Wood stood within a few feet of General O. O. Howard, when the latter was wounded in the arm which afterward had to be amputated. In the battle of Malvern Hill, the regiment was hotly engaged for several hours; but owing to its protected position, the loss sustained by the regiment was not very severe. Soon after the battle of Malvern Hill, Mr. Wood was severely injured while working on the fortifications, and after lying in the hospital for over six months, and not recovering, he was discharged from the army.

Seven months after his return from the war, Mr. Wood entered the Literary Department of Michigan University. At the commencement exercises of the University, three years after his matriculation, he was among those chosen to deliver orations. Of Mr. Wood's oration, the Detroit *Tribune* spoke as follows: "His oration was one of the best of the day, both as to literary and elocutionary merit. Some portions were of unusual beauty, and the delivery was emphatic and impressive."

During his collegiate years, he spent the major portion of his time at Latin and Greek, as he found a thorough knowledge of these languages very difficult to acquire. For him, Mathematics always was "an easy study"—a *delightful* study; and for the Natural Sciences, he had (and still has) a peculiar *fondness*. On graduation he received the Degree of *Bachelor of Arts* (A. B.), in 1866; subsequently, the Degree of *Master of Arts* (A. M.); and last June, from New Windsor College, the Degree of *Doctor of Philosophy* (Ph. D.)

Mr. Wood was married to Miss Mary Hicks, near Rochester, New York, September 2, 1868; and he has two sons, 18 and 20 years of age, who are attending the Stevens Institute of Technology;

After graduation, Professor Wood was the Principal of an Academy near Philadelphia, Pa., for eight years, when he accepted the position of Vice Principal and Professor of Higher Mathematics and Astronomy in the *Keystone State Normal School* of Pennsylvania. During his connection with this School, Professor Wood edited the *Scientific Department*, and subsequently the *Mathematical Department*, in the NATIONAL EDUCATOR.

Among his pupils at this Institution, there was a rather slender, fair-faced, and affable *Pennsylvania-German* youth who had taken his Degree in the Pedagogical Course, during June of the same year in which Professor Wood, in August, entered upon his duties as Vice-Principal and Professor of Mathematics and Astronomy. This youth had returned to his *Alma Mater*, in order to take his Degree in the Scientific Course, two years later. He was the only student in the Scientific Course. Being an industrious student with a mathematico-scientific bent of mind, this youth soon had gained the friendship of Professor Wood. Like *father* and *son*, the professor and the youth enthusiastically studied the

mathematical works of Loomis, Olney, Quinby, Courtenay, Bartlett, Todhunter—and even selections from the astronomical works of Chauvenet and Watson, for two long but profitable years. At the expiration of the second year, Professor Wood had the good fortune to see his youthful pupil (F. P. Matz) *passed* by the State Board of Examiners, and *graduated*, with "the highest distinction."

Afterwards Professor Wood held (for six years) a position in an educational institution in New York City, and subsequently was for three years the professor of Mathematics in the *Case School of Applied Science*, Cleveland, Ohio. From this last-named School, he was called to his present position, in 1888.

Dr. Wood is fond of Mathematics; and during the last twenty years, he has contributed articles and solutions of problems to *many* periodicals. Of late years, he has confined his attention more particularly to the works he is preparing for publication. His work, *Short Cuts and Curiosities in Mathematics*, is now passing through the press; and before the expiration of the current year, the American Book Company will have published his *Treatise on Plane and Spherical Trigonometry*. His *Perpetual Calendar*, good for ten centuries, has been pronounced the most unique calendar ever published. His article on *Method of Finding the Date of Easter*, has been highly commended. His *New Method of Extracting the Cube Root*, recently printed in the STEVENS INDICATOR, has been copied by numerous periodicals.

Dr. Wood has not confined his attention exclusively to Mathematics. He is well versed in the classics, well read in history, and an adept in geology. He is, also, an interesting speaker, and has delivered many public lectures illustrated with the stereopticon. His illustrated lectures on the Civil War are especially interesting.

As a teacher, Dr. Wood is earnest, untiring in his efforts, and patient to render assistance to those who acquire knowledge slowly. He is naturally a *leader*, and inspires his pupils with his own enthusiasm. He is the personification of kindness; but when he has to *drive*, he drives with an energy that is speedily satisfactory to those driven.

When Professor Wood left Cleveland, Ohio, in 1888, the following is an extract of what appeared in one of the large dailies of that city: "Professor H. A. Wood has become so well known in this city, and so highly esteemed by all who know him, that his contemplated change of residence will be felt as a great loss. He has made himself felt in the community as one always ready to do good. He has ever been formost in Sunday School work, mission enterprises, in church and social life, and in temperance and other reforms."

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from November Number.]

SCHOLION III, *in which is weighed the attempt of the Arab Nassaradin, and likewise the idea of the illustrious John Wallis upon the same affair.*

This endeavor of the Arab Nassaradin the already eulogised John Wallis has published in the Latin language with remarks added in opportune place.

However Nassaradin requires two things to be conceded to him in this affair.

The first is; that any two straight lines lying in the same plane, upon which ever-so-many other straight lines so strike, that they are always perpendicular to one indeed of these, but always cut the other at unequal angles, truly toward one part always under an acute angle, and toward the other always under an obtuse angle; that, I say, the above mentioned lines be supposed always more (as long as they do not mutually cut) to approach each other toward the side of those acute angles; and on the other hand always more to recede from one another toward the parts of the obtuse angles.

But I indeed, if nothing else impedes Nassaradin, willingly permit what he postulates; since just that, which with him remains undemonstrated can be recognized as most rigorously demonstrated by me in Cor. II. after P. III.

The other postulate of Nassaradin is the reciprocal of the first; that indeed the angle may always be acute toward those parts where the just mentioned perpendiculars are supposed to become always shorter; but obtuse toward the other parts where these perpendiculars are supposed to go out always longer. But here lurks an ambiguity.

For why (while from any one perpendicular prescribed as the first we proceed to the others) may not the angles of the consequent perpendiculars, on the same side acute, not become even greater, even to where one strikes upon a right angle, consequently upon such a perdendicular as is itself the common perpendicular to each of the aforesaid straights? And if indeed that happens, evanishes this subtle preparation of Nassaradin, by means of which ingeniously indeed, but with great labor he demonstrates the Euclidean postulate.

And yet if Nassaradin with a certain justice may determine to presume as if known 'per se' that persistence of acute angles on the same side: why can not also (I speak with Wallis) be assumed as if clear 'per se': *Two straights in the same plane converging* (upon which of course an other straight striking makes toward the same parts two angles less than two right angles, as suppose one right, and the other in whatever way acute) *finally meet, if produced?*

Nor in fact can it be objected, that this greater convergence toward one

side can always subsist within a certain determinate limit, so that indeed a certain so much of distance always intervenes between these lines on this side, even if still one approaches always more nearly to the other.

That cannot, I say, be objected ; since from this itself I will demonstrate, after P. XXV., the meeting at a finite distance of all such straights, in accordance with the Euclidean postulate.

Now I go over to the aforesaid John Wallis, who, as made a custom with so many great men, ancient as well as recent, and on the other hand from the obligation imposed on his Oxford professional chair, determined to undertake this same duty of demonstrating the oft mentioned postulate.

Now solely he assumes as if certain, what follows : namely that *to any given figure another similar of any magnitude is possible.*

And that this indeed may be presumed of any figure (although in his affair he assumes solely a rectilineal triangle) is well argued from the circle, which of course all admit can be described with any sized radius.

Further the acute man observes most cautiously it does not thwart this his presumption, that besides the equality of corresponding angles also the proportionality of all corresponding sides is required, in order that a rectilineal figure, for example a triangle, may be similar to another rectilineal triangle ; though still the definition of proportion, and forthwith of similar figures are to be taken from the fifth, and the sixth book of Euclid : *For* (says he himself) *Euclid could have put each in front of book first.*

Hereafter, this standing (which nevertheless can be denied by any one, unless it is demonstrated) the famous man carries out his intent with really beautiful and ingenious effort.

But I am unwilling to fail in anything to the charge undertaken by me.

Therefore I assume two triangles, one ABC, and the other DEF (fig. 24) mutually equiangular. I do not say wholly similar; because I do not need the proportionality of the sides about the equal angles, nay nor any determinate measure of the sides themselves. Merely therefore I do not wish triangles mutually equilateral, since then the eighth of book first would alone suffice, without any assumption.

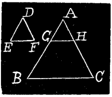


Fig. 24.

So let the angles at the points A, B, C, be equal to the angles at the points D, E, F; and let the side DE be less than the side AB; and in AB is assumed the portion AG equal to this DE, and likewise in AC the portion AH equal to this DF. But that DF must be less than AC I will make clear below. Then (GH joined) follows (from Eu. I. 4) the angles at the points E, and F will be equal to AGH, AHG. However since the just mentioned angles, together with the others BGH, CHG, are equal (Eu. I. 13) to four right angles ; likewise will be equal to four right angles the angles at the points B, and C, together with these same angles BGH, CHG. Therefore the four angles of the quadrilateral $BGHC$ will be together equal to four right angles ; and conse-

quently (from P. XVI.) is established the hypothesis of right angle; and at the same time (from P. XIII.) the Euclidean postulate.

Moreover I have supposed the side DF, or AH assumed equal to it, to be less than the side AC. For if it were equal, and so the point H should fall upon the point C, then the angle BCA would be equal (by hypothesis) to the angle EFD, or GCA (which then it would become) a part to the whole; which is absurd.

But if it were greater, and so the join GH should cut BC itself in some point, now the external angle ACB would be from the hypothesis equal (against Eu. I. 16) to the internal and opposite angle (which then would become) AHG, or GHA.

Therefore I have rightly supposed the side DF of one triangle to be less than the side AC of the other triangle, in accordance with the hypothesis now established.

Wherefore from any two triangles mutually equiangular, but not also mutually equilateral, the Euclidean postulate is established.

Quod intendebatur.

[To be Continued.]

HISTORICAL SURVEY OF THE ATTEMPTS AT THE COMPUTATION AND CONSTRUCTION OF π.

By DAVID EUGENE SMITH, Ph. D., Professor of Mathematics in the Michigan State Normal School, Ypsilanti, Michigan.

[NOTE. The following article is translated (by permission) from Professor Klein's recent work, *Vortraege ueber ausgewaehlte Fragen der Elementargeometrie, ausgearbeitet von F. Taegert*, Leipzig, Teubner, 1895. The work can not be too highly commended to teachers, since it is one of those exceedingly rare treatises in which a master of modern mathematics has treated elementary subjects from his high point of view.

Michigan State Normal School, December, 1895.]

Later in this work it will be proved that the number π belongs to that class of numbers known as transcendent, whose existence was shown in the preceding chapter. This fact was first proved by Lindemann in 1882, and a problem was thereupon settled which, so far as our information extends, has occupied the attention of mathematicians for 4000 years, namely, that of the quadrature of the circle.

It is evident that if the number π is not algebraic it cannot be constructed by means of the compasses and ruler. Hence the quadrature of the circle is, in the sense understood by the ancients, impossible. It is of greatest interest to follow the fortunes of this problem in the various epochs of Science, as ever new attempts were made to find a solution by means of the ruler and the

compasses, and to see how these necessarily fruitless attempts nevertheless worked for advancement in the manifold realm of mathematics.

The following brief historical survey is based upon Rudio's excellent treatise, *Archimedes, Huygens, Lambert, Legendre; Vier Albandlungen ueber die Kreismessung, Leipzig*, 1892. In this work are given in German translation the contributions of the writers named. Even though the presentation of the matter is remote from the more modern methods here discussed,* nevertheless it includes many very interesting details which are of especial value in elementary teaching.

1. Among the attempts to determine the ratio of the diameter to the circumference we may first distinguish the empirical stage in which it was sought to attain the desired end through measuring or estimating. The oldest known mathematical work, the Rhind Papyrus (c. 2000 B. C.) contains the problem in the well-known form, to transform a circle into a square of equal area. The writer of the papyrus, Ahmes, lays down the following rule: Cut off $\frac{1}{9}$ of a diameter and construct a square on the remainder; this has the same area as the circle. The value of π thus obtained is $(\frac{16}{9})^2=3.16\ldots\ldots$, not very inexact. Still farther from the correct value is that of $\pi=3$ which is found in the Bible. (I Kings, 7:23, and II Chron. 4:2.)

2. The Greeks raised themselves above this impirical standpoint, and especially Archimedes, who in his work *Κύκλου μέτρησις* computes the area of the circle by the help of inscribed and circumscribed polygons, as is still done in the schools. His method remained in use until the invention of the differential calculus, and was extended and made practically usable especially by Huygens (†1654) in his work *De circuli magnitudine inventa*.

As in the case of the duplication of the cube and the trisection of an angle the Greeks then sought to attain the quadrature of the circle by the help of higher curves.

We may, for example, consider the curve, $y=\text{arc}\sin x$ [usually written in English $y=\sin^{-1}x$; the Continental form will be followed in this translation] which represents the curve of sines placed vertically. Geometrically, π appears as a special ordinate of this curve, analytically as a special value of our transcendent function. Apparatus which describes transcendent curves we will call transcendent apparatus. A piece of transcendent apparatus which draws the curve of sines gives us a real construction for π. The curve $y=\text{arc}\sin x$ we designate now-a-days as an *integral curve*, because it can be defined by means of the integral of an algebraic function: $y=\int\frac{dx}{\sqrt{1-x^2}}$. The ancients called such a curve a Quadratrix or *τετραγωνίζουσα*. The best known of these is the Quadratrix of Dinostratus (c. 350 B. C.) which, however, had been already constructed by Hippias of Elis (c. 420 B. C.) for the trisection of an angle. It may be geometrically defined as follows: On the line OB and the arc AB two points, M and L, move

*In a note to the translator Professor Klein says: "This remark concerning Rudio's work is not happily expressed. The meaning is not that modern researches, so far as then carried, are not given in the work, but they are not deduced."

with uniform velocity. They start at the same time from O and A, respectively, and they reach B at the same time. If OL is drawn, and through M the parallel to OA which meets OL at P, then P is a point of the Quadratrix. From this definition it follows that y and θ are proportional. Further, since for $y=1$, $\theta=\frac{\pi}{2}$, we have $\theta=\frac{\pi}{2}y$, and from $\theta=\text{arc}\tan\frac{y}{x}$ the equation of the curve becomes $\frac{y}{x}=\tan\frac{\pi}{2}y$. The point in which the line cuts the x-axis will be found from $x=\frac{y}{\tan\frac{\pi}{2}y}$ if y becomes 0. Since for small values the tangent equals its argument, it follows that $x=\frac{2}{\pi}$. Hence the radius of the circle is the mean proportional between the quadrant of the circle and the abscissa of the point of intersection of the Quadratrix with the x-axis. The Quadratrix can, therefore, be used in the rectification problem, and hence for the quadrature of the circle. Fundamentally, however, the curve is only a geometric formulation of the rectification problem, that is so long as no apparatus is given by which it can be described by a continuous line.

3. The rise of modern analysis occurs in the period from 1670 to 1770, a period characterized by the names of Leibniz, Newton, and Euler. In the midst of so many great discoveries following closely on one another, it is natural that strict criticism took a somewhat backward step. Among these discoveries is one of especial concern to us, the development of the theory of series. Especially for π were a great number of approximations brought forward, of which we may mention only the so-called Leibniz series (which, however, was known before Leibniz): $\frac{\pi}{4}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots$ Furthermore this period brings the discovery of the connection between e and π. The number e and the natural logarithms and with them the exponential function are first found *in embryo* in the works of Napier (1614). This number seemed at first to have no relation to the circular functions and to the number π, until Euler had the courage to attack the problem by means of imaginary exponents. In this way he reached the celebrated formula $e^{ix}=\cos x+i\sin x$, which for $x=\pi$ becomes $e^{i\pi}=-1$. This formula is without doubt one of the most notable of all mathematics. With it are connected the modern proofs of the transcendence of π since they first show the transcendence of e.

4. After 1770 criticism again took the upper hand. In 1770 appeared Lambert's work, *Vorläufige Kentnisse für die, so die Quadratur des Cirkuls suchen.* He treated there and elsewhere the irrationality of π. In 1794 Legendre showed conclusively in his *Éléments de Géométrie* that π and π^2 are irrational numbers.

5. But it was not until a hundred years later than this that modern research began. The starting point of this research is the work of Hermite, *Sur la fonction exponentielle* (*Compt. Rend.* 1873, published separately in 1874). In this

is proved the transcendence of e. Closely following Hermite came the same proof for π by Lindemann in a dissertation *Ueber die Zahl* π (*Math. Ann.* 20, 1882. See also the proceedings of the Berlin and Paris academies). With this the matter was now for the first time settled, nevertheless the treatment given by Hermite and Lindemann is very complicated.

The first simplification was given by Weierstrass in the *Berliner Berichte* in 1885. The above mentioned works Bachman embodied in his text-book, *Vorlesungen ueber die Natur der Irrationalzahlen*, 1892.

The spring of 1893 brought, however, new and very important simplifications. In the first rank should be named the developments of Hilbert in the *Göttinger Nachrichten.* Hilbert's proof is not wholly elementary; it contains still a remnant of Hermite's course of reasoning in the integral

$$\int_0^\infty z^\rho e^{-z} dz = \rho\,!.$$

But Herwitz and Gordan soon after showed that this transcendental part might be eliminated. (*Göttinger Nachrichten* and *Comptes Rendus* respectively; all three dissertations are reproduced in the *Math. Annalen*, Bd. 43, either literally or somewhat extended). So the matter has now become so elementary that it is generally available.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from November Number.]

The Construction of Non-Primitive Groups with Two Systems of Non-Primitivity.

Let the degree of the required non-primitive group be $2n$, and consider the $(n\,!)^2$ substitutions

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all } a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n$$

and also the group of order $2(n\,!)^2$

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n).$$

The latter is clearly a non-primitive group of degree $2n$ and the former are the substitutions of this group which interchange the systems. It is easily seen

that G_1 can have no larger value than it has in the above non-primitive group, and that every G_1 for other non-primitive groups may be regarded as a subgroup of this G_1. From this it follows that the first set of substitutions includes all the substitutions which can be used with any G_1 to form a non-primitive group, for if there were such a substitution s_x which is not in the first set then we would obtain more than $(n!)^2$ different substitutions which transform

$$(a_1 a_2 \ldots\ldots a_n)\mathrm{all}(b_1 b_2 \ldots\ldots b_n)\mathrm{all}$$

into itself without interchanging the systems by multiplying one substitution of this set into the entire set increased by s_x. Hence all the substitutions which can be used to interchange the systems are found in the first set. In a similar way we can show that the number of the snbstitutions which interchange the systems must *always* be equal to the order of G_1. Hence if in any non-primitive group we represent the substitution which interchange the systems by G_2 and the non-primitive group by G we have

$$G = G_1 + G_2$$

where G_1 and G_2 contain the same number of substitutions and G is a subgroup of

$$(a_1 a_2 \ldots\ldots a_n)\mathrm{all}(b_1 b_2 \ldots\ldots b_n)\mathrm{all}(a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n).$$

Suppose any G_1 constructed by combining a transitive* subgroup of $(a_1 a_2 \ldots\ldots a_n)$all with a conjugate subgroup of $(b_1 b_2 \ldots\ldots b_n)$all and suppose s_y to have the following properties :

(1) its square is found in G_1;

(2) it transforms G_1 into itself ;

(3) it interchanges the systems of G_1.

Then will all of the substitutions

$$G_1 s_y = G_2$$

*We shall henceforth assume that the systems of non-primitivity are the transitive constituents of G_1. We proved above that this can always be done but we did not prove that it is possible to regard intransitive constituents of G_1 as systems. That this may be done is proved by the following instance in which

$$G_1 = (ab.cd.ef.gh.ij.kl)$$

and the systems are either *a,b; c,d; e,f; g,h; i,j;* and *k,l* or *a,b,c,d; e,f,g,h; i,j,k,l.* Letting the letters *A, B,* etc., stand for the first systems and *A', B', C'* for the second we may write the group as follows:

$$\begin{array}{ll}
\left.\begin{array}{l} 1 \\ ab.cd.ef.gh.ij.kl \end{array}\right\} & 1 \\
\left.\begin{array}{l} aei.bfj.cgk.dhl \\ afibej.chkdgl \end{array}\right\} & \begin{array}{l} ABC.DEF \\ A'B'C' \end{array} \\
\left.\begin{array}{l} aie.bjf.ckg.dlh \\ ajebif.clgdkh \end{array}\right\} & \begin{array}{l} ACB.DFE \\ A'C'B' \end{array} \\
\left.\begin{array}{l} ac.bd.ek.fl.gi.hj \\ ad.bc.el.fk.gj.hi \end{array}\right\} & \begin{array}{l} AD.BF.CE \\ B'C' \end{array} \\
\left.\begin{array}{l} ag.bh.ce.df.ik.jl \\ ah.bg.cf.de.il.jk \end{array}\right\} & \begin{array}{l} AE.BD.CF \\ A'B' \end{array} \\
\left.\begin{array}{l} ak.bl.ci.dj.eg.fh \\ al.bk.cj.di.eh.fg \end{array}\right\} & \begin{array}{l} AF.BE.CD \\ A'C' \end{array}
\end{array}$$

If we consider the six systems they are the transitive constituents of G_1 but if we consider only the three systems they are intransitive constituents.

have these properties and it can be easily seen that $G_1 + G_2$ constitute a non-primitive group. Hence it follows that it is only necessary to find one substitution which possesses the three properties named above in order to obtain a G_2 corresponding to a given G_1.

To fix these ideas we proceed to find all the non-primitive groups whose degree does not exceed six. Since n must be the degree of some group it follows that $2n$ cannot be less than four.

Non-Primitive Groups of Degree Four.

G_1 must be either $(ac.bd)$ or $(ac)(bd)$. G_2 is found in $(ac)(bd)ab.cd = ab.cd$, $abcd$, $adcb$, $ad.bc$. If $G_1 = (ac.bd)$ we see at once that $ab.cd$ and $abcd$ satisfy the three required conditions. In the first case $G_2 = ab.cd$, $ad.bc$ and in the second case it equals $abcd$, $adcb$. Hence the given G_1 leads to the following two non-primitive groups of degree and order four : (A transitive group is called *regular* when its degree is equal to its order.)

$$(abcd)_4,\ (abcd)\text{cyc}.$$

If $G_1 = (ac)(bd)$ we see again directly that $ab.cd$ satisfies the three required conditions, as we found in the general case. The corresponding G_2 includes all the possible substitutions. We obtain therefore only one non-primitive group with this G_1, viz :

$$(abcd)_8$$

Hence there are *three* non-primitive groups of degree four. The other two transitive groups of degree four are multiply transitive and therefore primitive.

Non-Primitive Groups of Degree Six with Two Systems of Non-Primitivity.

G_1 must be one of the following five groups :

$$(abc)\text{all}(def)\text{all},\ \{(abc)\text{all}(def)\text{all}\}\text{pos},\ (abc.def)\text{all}$$
$$(abc)\text{cyc}(def)\text{cyc},\ (abc.def)\text{cyc}.$$

G_2 is found in

$$(abc)\text{all}(def)\text{all}\ ad.be.cf.$$

(*a*) If $G_1 = (abc)\text{all}(def)\text{all}$, G_2 will include all the possible substitutions and we obtain one group of order 72, viz :

$$(1) \qquad (abc)\text{all}(def)\text{all}(ad.be.cf).$$

(*b*) If $G_1 = \{(abc)\text{all}(def)\text{all}\}\text{pos}$ the two substitutions $ad.be.cf$ and $aebd.cf$ satisfy the three conditions and we thus obtain one G_2 which contains only negative substitutions and another which contains only positive substitutions. The two resulting groups are

(2) $\{(abc)\text{all}(def)\text{all}\}\,\text{pos}(ad.be.cf)=(abcdef)_{36_3}$ *

(3) $\{(abc)\text{all}(def)\text{all}\}\,\text{pos}(aebd.cf)=(abcdef)_{36}$.

(c) If $G_1=(abc.def)\text{all}\ ad.be.cf$ satisfies the three necessary conditions. We thus obtain

(4) $(abc.def)\text{all}(ad.be.cf)=(abcdef)_{12}$.

No substitutions except those in the above G_2 can transform $(abc.def)$all into itself and interchange the systems, because no two substitutions of (abc)all transform (abc)all in the same way. Hence there is only one G_2 for the given G_1.

(d) If $G_1=(abc)\text{cyc}(def)\text{cyc}$, then the square of only half of the substitutions in which G_2 is found are contained in this G_1. Hence two G_2's are possible, viz:

$$G_1\ ad.be.cf \text{ and } G_1\ ab.de.ad.be.cf=G_1\ ae.bd.cf.$$

ab transforms G_1 into itself and one of these G_2's into the other so that there is really only the following non-primitive group with the given G_1†:

(5) $(abc)\text{cyc}(def)\text{cyc}(ad.be.cf)$.

(e) Finally if $G_1=(abc.def)\text{cyc}$ we obtain two G_2's and hence the following groups:

(6) $(abc.def)\text{cyc}(ad.be.cf)=(abcdef)\text{cyc}$

(7) $(abc.def)\text{cyc}(ae.bd.cf)=(abcdef)_6$

The first one of these two will be found in three conjugate forms if we use all the possible G_2's. We have now examined all the possible G_1's and found seven non-primitive groups of degree six which contain two systems of non-primitivity.

[To be Continued.]

*This group is not found in Professor Cayley's list, *Quarterly Journal of Mathematics*, Vol. 25, pp. 71—79. It is found in Professor Cole's supplementary list, *Bulletin of the New York Mathematical Society*, May, 1893.

†It has been proved that whenever G_1 is the product of two groups then there is really only one G_2 for the given G_1. We shall give a proof of this theorem later.

THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

[Continued from September-October Number.]

IV. Typical of the Cassinian Oval, we have by the *Method of Complex Variables* the following equations:

$$x=\beta\sqrt{(1+\alpha t)},=\beta\sqrt{[1+\alpha(\cos\theta+i\ \sin\theta)]}\ldots\ldots(A);$$

$$y=\beta\sqrt{(1+\alpha/t)},=\beta\sqrt{[1+\alpha(\cos\theta-i\ \sin\theta)]}\ldots\ldots(B).$$

$$\therefore\ dx/d\theta=\tfrac{1}{2}i^2\alpha\beta(\cos\theta-i\ \sin\theta)/\sqrt{[1+\alpha(\cos\theta+i\ \sin\theta)]}\ldots\ldots(A');$$

$$dy/d\theta=\tfrac{1}{2}i^2\alpha\beta(\cos\theta+i\ \sin\theta)/\sqrt{[1+\alpha(\cos\theta-i\ \sin\theta)]}\ldots\ldots(B').$$

$$\therefore\left(\frac{d\mathbf{P}_1}{d\theta}\right)^2=\frac{i^4\alpha^2\beta^2(\cos\theta-i\ \sin\theta)(\cos\theta+i\ \sin\theta)}{4\sqrt{\{[1+\alpha(\cos\theta+i\ \sin\theta)]\times[1+\alpha(\cos\theta-i\ \sin\theta)]\}}}\ldots\ldots(C).$$

$$\therefore\ \mathbf{P}=\alpha\beta\int_0^{2\pi}\frac{d\theta}{[1+\alpha^2+2\alpha\cos\theta]^{\frac{1}{2}}}=\frac{2\alpha\beta}{[1+\alpha^2]^{\frac{1}{2}}}\int_0^{\pi}\frac{d\theta}{[1+M\cos\theta]^{\frac{1}{2}}}\ \ldots\ldots(D).$$

[From (A) and (B), after differentiating with respect to t, we have

$$dx/dt=\tfrac{1}{2}\alpha\beta/x=\tfrac{1}{2}\alpha\beta/\sqrt{(1+\alpha t)},$$

$$dy/dt=-\tfrac{1}{2}\alpha\beta/t^2y=-\tfrac{1}{2}\alpha\beta/t^2\sqrt{(1+\alpha/t)}.$$

$$\therefore\ (d\mathbf{P}_1)^2=\left(\frac{-\alpha^2\beta^2}{4\sqrt{[(1+\alpha t)(1+\alpha/t)]}}\right)\left(\frac{dt}{t}\right)^2\ldots\ldots(E).$$

Differentiating under the assumption that $t=\cos\theta+i\ \sin\theta$, etc.,

$$\frac{dt}{t}=\left(\frac{i\ \cos\theta-\sin\theta}{\cos\theta+i\ \sin\theta}\right)d\theta=i\left(\frac{\cos\theta+i\ \sin\theta}{\cos\theta+i\ \sin\theta}\right)d\theta=id\theta.$$

$$\therefore\ \mathbf{P}_1=\tfrac{1}{2}i\alpha\beta\int\frac{dt}{t[1+\alpha^2+\alpha(t+1/t)]^{\frac{1}{2}}}=\tfrac{1}{2}\alpha\beta\int\frac{d\theta}{[1+\alpha^2+2\alpha\cos\theta]^{\frac{1}{2}}};$$

$$\text{and } \mathbf{P}=\tfrac{1}{2}\alpha\beta\int_0^{4\pi}\frac{d\theta}{[1+\alpha^2+2\alpha\cos\theta]^{\frac{1}{4}}}=\frac{2\alpha\beta}{[1+\alpha^2]^{\frac{1}{4}}}\int_0^{\pi}\frac{d\theta}{[1+M\cos\theta]^{\frac{1}{4}}}\,.\Big]$$

Since in the Cassinian Oval under consideration, $\alpha=\frac{5}{4}$ and $\beta=2$, we have $M=\frac{40}{41}$; that is, (D) is *identical* with (4) on page 265 of the September-October MONTHLY. Slowly converging series may be obtained by transforming under the hypothesis that $\theta=(90-\phi)$, or under the hypothesis that $\theta=(90+\phi)$.

V. The assumption of (2) from page 264 of the MONTHLY specified gives

$$\mathbf{P}=m^2\int_b^a\frac{r^2dr}{\sqrt{\{[(m^2+c^2)^2-r^4]\times[r^4-(m^2-c^2)^2]\}}}\;\ldots\ldots(2).$$

Let $(m^2-c^2)/(m^2+c^2)=e^2$, and $r^2=(m^2+c^2)x^2\ldots\ldots(k)$;

$$\text{then } \mathbf{P}=\frac{4m^2}{\sqrt{(m^2+c^2)}}\int_e^1\frac{2(m^2+c^2)^2x^2dx}{\sqrt{\{[(m^2+c^2)^2(1-x^4)]\times[(m^2+c^2)^2(x^4-e^4)]\}}}\ldots.(\delta),$$

$$\text{or } \mathbf{P}=\frac{4m^2}{\sqrt{(m^2+c^2)}}\int_e^1\frac{2x^2dx}{\sqrt{[(1-x^4)(x^4-e^4)]}}\;\ldots\ldots(F),$$

$$=\frac{4m^2}{\sqrt{(m^2+c^2)}}\Big[\int_e^1\frac{(x^2-e)dx}{\sqrt{[(1-x^4)(x^4-e^4)]}}+\int_e^1\frac{(x^2+e)dx}{\sqrt{[(1-x^4)(x^4-e^4)]}}\Big]\ldots\ldots(G),$$

$$=\sqrt{\Big\{\Big(\frac{16m^4}{m^2+c^2}\Big)\Big(\frac{1}{2(1+e^2)}\Big)\Big\}}\Big[cn^{-1}\Big(\frac{x+e/x}{1+e},\ \frac{1+e}{\sqrt{[2(1+e^2)]}}\Big)$$

$$+\,cn^{-1}\Big(\frac{x-e/x}{1-e},\ \frac{1-e}{\sqrt{[2(1+e^2)]}}\Big)\Big]_1^e=\sqrt{\Big\{\Big(\frac{16m^4}{m^2+c^2}\Big)\Big(\frac{1}{2(1+e^2)}\Big)\Big\}}\Big[\{cn^{-1}(+1)$$

$$-cn^{-1}(+1)\}+\{cn^{-1}(-1)-cn^{-1}(+1)\}\Big]\ldots\ldots(H);$$

that is, the *first* indicated integral in (G) *has vanished.* The expression for the perimeter of the *Cassinian Oval*, therefore, becomes

$$\mathbf{P}=2\pi m\Big\{1+\Sigma\Big(\frac{1.3.5.7\ldots.(2n-1)}{2.4.6.8\ldots\ldots\ldots.2n}\Big)^2\Big[\sqrt{\Big(1+\frac{c^2}{m^2}\Big)}-\sqrt{\Big(1-\frac{c^2}{m^2}\Big)}\Big]^{2n}\Big\},$$

which is a *complete* elliptic integral of the *first* order.

For the perimeter of the *Bernoullian Lemniscate*, we have $m=c$; that is, symmetrically expressed,

$$\mathbf{P}'=2\pi c\Big\{1+\Sigma\Big(\frac{1.3.5.7\ldots..(2n-1)}{2.4.6.8\ldots\ldots\ldots.2n}\Big)^2\Big[\sqrt{\Big(1+\frac{c^2}{c^2}\Big)}-\sqrt{\Big(1-\frac{c^2}{c^2}\Big)}\Big]^{2n}\Big\}.$$

For the perimeter of the two *Ovaliform Figures*, we have $m\angle c$; that is, similarly expressed,

$$\mathbf{P}''\sqrt{2\pi c\left(\frac{m}{c}\right)^2}\left\{1+\Sigma\left(\frac{1.3.5.7\ldots\ldots(2n-1)}{2.4.6.8\ldots\ldots\ldots.2n}\right)^2\left[\sqrt{\left(1+\frac{m^2}{c^2}\right)}-\sqrt{\left(1-\frac{m^2}{c^2}\right)}\right]^{2n}\right\}.$$

[Concluded.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by D. P. WAGONER, A. B., Principal of the School of Language, Westerville, Ohio.

A man bought a farm for \$6000 and agreed to pay for it in four equal installments, at 6 per cent. annual interest compounded every instant. Required the aunual payment. *B. F. Burleson.*

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana, College, Texarkana, Arkansas-Texas; P. S. BERG, Larimore, North Dakota; and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let $a=\$6000$, $r=.06$, x=annual payment, and $m=4$=number of equal annual payments. If the interest is compounded n times a year, we have the present value of the first installment$=x(1+\frac{r}{n})^{-n}=xe^{-r}$ when n is infinite; of the second, $=xe^{-2r}$; of the third, $=xe^{-3r}$; of the mth, $=xe^{-mr}$; where e= Naperian base.

(See Todhunter's Differential Calculus, page 136).

$$\therefore\ a=x\left(\frac{1}{e^r}+\frac{1}{e^{2r}}+\frac{1}{e^r}+\ldots\ldots+\frac{1}{e^{mr}}\right)=\frac{x}{e^{mr}}\left(\frac{e^{mr}-1}{e^r-1}\right)$$

$$\therefore\ x=\frac{a\ e^{mr}(e^r-1)}{e^{mr}-1}=\frac{a(e^r-1)}{1-e^{-mr}}=\frac{a(e^r-1)}{1-e^{-4r}}$$

$$\therefore\ x=\$6000\left(\frac{e^{.06}-1}{1-e^{-.24}}\right)=\$1738.269.$$

II. Solution by B. F. BURLESON, Oneida Castle, New York.

The amount of P in n years at $r=6\%$ when the interest is compounded q times a year is evidently

$$A=P(1+\frac{r}{q})^{nq}\ldots\ldots(1).$$

Expanding the right hand member in (1) by the binomial theorem, we have

$$A=P(1+nq.\frac{r}{q}+\frac{nq(nq-1)}{2\,!}.\frac{n^2}{q^2}+\ldots\ldots(2).$$

When $q=\infty$, equation (2) becomes

$$A=P(1+nr+\frac{n^2r^2}{2\,!}+\frac{n^3r^3}{3\,!}+\text{etc.,})=,$$

by the exponential theorem, $Pe^{nr}\ldots\ldots(3)$.

Whence by taking logarithms in (3) and changing to the common system by multiplying by its modulus we have in inverse functions

$$A=P\log(1.4342944nr)=106.1836,$$

when $n=1$, $r=.06$, and $P=100$.

Having determined the rate in equivalent annual compound rate, the required annual payment is determined as follows :

$1\div1.061836=.941765=P$

$1\div1.061836^2=.886913=P'$

$1\div1.061836^3=.8352716=P''$

$1\div1.061836^4=.7866294=P'''$.

Now $\$6000\div(P+P'+P''+P''')=\$1738.834$.

This problem was also solved by *B. F. YANNEY*.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

50. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

Given $b=a\sqrt{-1}.\tan\frac{m\pi}{n}$, m being an arbitrary integer, find the simplest *algebraic* relation between a and b.

Solution by the PROPOSER.

From $b=ai\tan\frac{m\pi}{n}$, where $i=\sqrt{-1}$, we derive $\frac{a+b}{a-b}=\frac{1+i\tan\frac{m\pi}{n}}{1-i\tan\frac{m\pi}{n}}$

$$=\frac{1-\tan^2\frac{m\pi}{n}+2i\tan\frac{m\pi}{n}}{1+\tan^2\frac{m\pi}{n}}=\cos^2\frac{m\pi}{n}-\sin^2\frac{m\pi}{n}+2i\sin\frac{m\pi}{n}.\cos\frac{m\pi}{n}=\cos\frac{2m\pi}{n}$$

$+i\sin\frac{2m\pi}{n}$, the n^{th} power of which, by De Moivre's Theorem, equals 1. $\therefore (a+b)^n=(a-b)^n$.

Also solved by *F. P. MATZ.*

51. Proposed by J. W. NICHOLSON, LL. D., President, and Professor of Mathematics, Louisiana State University and A. and M. College, Baton Rouge, Louisiana.

Solve the equation $x^5+5mx^3+5m^2x+n=0$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let $x=y+z$. The equation then easily reduces to $y^5+z^5+5(yz+m)\{y^3+z^3+(2yz+m)(y+z)\}+n=0$. Now x may be decomposed into two parts, y and z, in an infinite variety of ways, and we may, therefore, suppose y and z are such as to satisfy the condition $yz+m=0$. This gives $yz=-m$, $y^5+z^5=-n$. Let $y^5=t_1$, $z^5=t_2$, then we have $t_1t_2=-m^5$, $t_1+t_1=-n$. $\therefore t_1$ and t_2 are the roots of the equation $t^2+nt-m^5=0$......(1). $\therefore x=y+z=\sqrt[5]{t_1}+\sqrt[5]{t_2}$.

Case I. When m is positive.

Let $t=u\sqrt{m^5}$, then (1) becomes $u^2+\frac{n}{\sqrt{m^5}}u-1=0$; but $\tan^2\frac{1}{2}\theta+2\cot\theta\tan\frac{1}{2}\theta-1=0$. $\therefore 2\cot\theta=n/\sqrt{m^5}$, or $\tan\theta=2\sqrt{m^5}/n$. $\therefore t_1=\sqrt{m^5}\tan\frac{1}{2}\theta$, $t_2=-\sqrt{m^5}\cot\frac{1}{2}\theta$. $\therefore x=\sqrt{m}(\sqrt[5]{\tan\frac{1}{2}\theta}-\sqrt[5]{\cot\frac{1}{2}\theta})$, where $\theta<90°$. Four of the five roots of $\sqrt[5]{\tan\frac{1}{2}\theta}$ are imaginary. Let $\tan\frac{1}{2}\phi=r=$the real value of $\sqrt[5]{\tan\frac{1}{2}\theta}$, and let a_1, a_2, a_3, a_4 represent the four imaginary roots of unity

$$\frac{\sqrt{5}-1+\sqrt{-10-2\sqrt{5}}}{4},\quad \frac{\sqrt{5}-1-\sqrt{-10-2\sqrt{5}}}{4},\quad \frac{-\sqrt{5}-1-\sqrt{-10+2\sqrt{5}}}{4},$$

$\frac{-\sqrt{5}-1+\sqrt{-10+2\sqrt{5}}}{4}$, respectively.

Then $x_1=\sqrt{m}(r-\frac{1}{r})$, $x_2=\sqrt{m}(ra_1-\frac{1}{ra_1})$, $x_3=\sqrt{m}(ra_2-\frac{1}{ra_2})$, $x_4=\sqrt{m}(ra_3=\frac{1}{ra_3})$, $x_5=\sqrt{m}(ra_4-\frac{1}{ra_4})$. Substituting the values of r, a_1, a_2, a_3, a_4,

we get $x_1=-2\sqrt{m}\cot\phi$,

$$x_2=-\tfrac{1}{2}\sqrt{m}\{(\sqrt{5}-1)\cot\phi-\sqrt{-10-2\sqrt{5}}\operatorname{cosec}\phi\},$$
$$x_3=-\tfrac{1}{2}\sqrt{m}\{(\sqrt{5}-1)\cot\phi+\sqrt{-10-2\sqrt{5}}\operatorname{cosec}\phi\},$$
$$x_4=\tfrac{1}{2}\sqrt{m}\{(\sqrt{5}+1)\cot\phi+\sqrt{-10+2\sqrt{5}}\operatorname{cosec}\phi\},$$
$$x_5=\tfrac{1}{2}\sqrt{m}\{(\sqrt{5}+1)\cot\phi-\sqrt{-10+2\sqrt{5}}\operatorname{cosec}\phi\}.$$

Case II. When m is negative and $-4m^5<n^2$.

Then (1) becomes $t^2+nt+(-m^5)=0$, or $u^2+\dfrac{n}{\sqrt{-m_5}}u+1=0$; but $\tan^2\frac{1}{2}\theta-2\operatorname{cosec}\theta\tan\frac{1}{2}\theta+1=0$. $\therefore -2\operatorname{cosec}\theta=n/\sqrt{-m^5}$, or $\sin\theta=-2\sqrt{-m^5}/n$.

$\therefore t_1=\sqrt{-m^5}\tan\frac{1}{2}\theta$, $t_2=\sqrt{-m^5}\cot\frac{1}{2}\theta$. $\therefore x=\sqrt{-m}(\sqrt[5]{\tan\frac{1}{2}\theta}+\sqrt[5]{\cot\frac{1}{2}\theta})$.

By a process similar to that in Case I, we get,

$$x_1=\sqrt{-m}(r+\tfrac{1}{r})=2\sqrt{-m}\operatorname{cosec}\phi,$$
$$x_2=\sqrt{-m}(ra_1+\tfrac{1}{ra_1})=\tfrac{1}{2}\sqrt{-m}\{(\sqrt{5}-1)\operatorname{cosec}\phi-\sqrt{-10-2\sqrt{5}}\cot\phi\},$$
$$x_3=\sqrt{-m}(ra_2+\tfrac{1}{ra_2})=\tfrac{1}{2}\sqrt{-m}\{(\sqrt{5}-1)\operatorname{cosec}\phi+\sqrt{-10-2\sqrt{5}}\cot\phi\},$$
$$x_4=\sqrt{-m}(ra_3+\tfrac{1}{ra_3})=-\tfrac{1}{2}\sqrt{-m}\{(\sqrt{5}+1)\operatorname{cosec}\phi-\sqrt{-10+2\sqrt{5}}\cot\phi\},$$
$$x_5=\sqrt{-m}(ra_4+\tfrac{1}{ra_4})=-\tfrac{1}{2}\sqrt{-m}\{(\sqrt{5}+1)\operatorname{cosec}\phi+\sqrt{-10+2\sqrt{5}}\cot\phi\}.$$

Case III. When m is negative and $-4m^5<n^2$.

In this case the preceding method fails. Let $x=ku$, then $x^5+5mx^3+5m^2x+n=0$ becomes $u^5+\dfrac{5mu^3}{k^2}+\dfrac{5m^2u}{k^4}+\dfrac{n}{k^5}=0$; also $\cos^5\theta-\frac{5}{4}\cos^3\theta+\frac{5}{16}\cos\theta-\frac{1}{16}\cos5\theta=0$. Let $u=\cos\theta$, then $5m/k^2=-5/4$,

$$\therefore k=2\sqrt{-m},\ -\tfrac{1}{16}\cos5\theta=\frac{n}{k^5}=\frac{n}{32m^2\sqrt{-m}},\quad \therefore \cos5\theta=-\frac{n}{2m^2\sqrt{-m}}.$$

Since $-m$ is positive, this gives five real values of θ, to be taken $<90°$.

$$\therefore x_1=2\sqrt{-m}\cos\theta,\ x_2=2\sqrt{-m}\cos(\frac{2\pi}{5}-\theta),\ x_3=2\sqrt{-m}\cos(\frac{2\pi}{5}+\theta),$$
$$x_4=2\sqrt{-m}\cos(\frac{4\pi}{5}-\theta),\ x_5=2\sqrt{-m}\cos(\frac{4\pi}{5}+\theta).$$

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

Let $x=z-\dfrac{m}{z}$, and the equation reduces to $z^{10}+nz^5=m^5$.

Whence $z=\sqrt[5]{-\dfrac{n}{2}\pm\sqrt{4m^5-n^2}}$.

III. **Solution by H. C. WILKES, Skull Run, West Virginia; and A. H. BELL, Hillsborough, Illinois.**

Factoring, etc., $(x^2+2m)(x^2+3m)=m^2-\frac{n}{x}$. Let $x=n$; then $x=n$, $\pm\sqrt{-m-1}$, $\pm\sqrt{1-2m}$, which will be the five roots.

Or $(x^2+3m)(x^2+2m)=(m+\sqrt{\frac{n}{x}})(m-\sqrt{\frac{n}{x}}$. Assuming $x^3+3m=\sqrt{\frac{n}{x}}$, $x^3+2m=m-\sqrt{\frac{n}{x}}$. $x=\sqrt{-\frac{3m}{2}}$, $m=2\sqrt{\frac{n}{x}}$; hence $x=\frac{4n}{m^2}$. Substituting $m=2\sqrt{\frac{n}{x}}$ for m in eq. 1, $x^5+10x^3\sqrt{\frac{n}{x}}+21n=0$. This can be developed, $x^{10}-58nx^5+441=0$. $\therefore x=\sqrt[5]{49n}$ or $\sqrt[5]{9n}$.

[The above is not strictly a solution, but affords a method of discovering integer roots, if any. The solu- of Professor Zerr is especially full and neat. EDITOR.]

Also solved by *F. P. MATZ.*

52. **Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

In how many ways can we arrange 12 friends of the MONTHLY, around a table, so that; (1) the editors may never be together, (2) Matz and Halsted may never be apart, and (2) Zerr and Ellwood may always have Gruber betwixt them?

Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

I. Considering one editor in position the other may occupy 9 places; but the first editor may take 12 places, and therefore the two take 108 positions. For each of these places the remaining nine mathematicians may be seated in $\underline{|9}$ ways, making $108\underline{|9}$ ways altogether.

II. If Matz and Halsted are never apart we may consider them as an element to be arranged as *each* of the other individuals. We then have $\underline{|11}$ ways of arranging them without regarding the *internal* arrangement of the group; this may be arranged in two ways. We, therefore, have $2\underline{|11}$ as the number of arrangements.

III. By the same reasoning as in the last case we have the number of arrangements$=2\underline{|10}$.

NOTE.—No solution of problem 53 has as yet been received. The published solution of problem 49, in last issue, should have been credited to Prof. J. H. Grove, Howard Payne College, Brownwood, Texas.

PROBLEMS.

59. **Proposed by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.**

Demonstrate the identity $2^{2n+1}\frac{d^n}{dx^n}\left(x^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}\right)=e^{\sqrt{x}}$.

60. Proposed by Professor C. E. WHITE, Trafalgar, Indiana.

Prove that every algebraic equation can be transformed into another equation of the same degree, but which wants its n^{th} term.

61. Proposed by J. A. CALDERHEAD, A. B., Superintendent of Schools, Limaville, Ohio.

Given $x^2+x\sqrt{xy}=10$, and $y^2+y\sqrt{xy}=20$ to find x and y by quadratics.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by F. P. MATZ, D. Sc., Ph,D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The closed portion of the curve known as "The Cocked Hat," equation

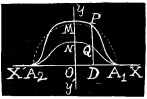
$$x^4+x^2y^2+4ax^2y-2a^2x^2+3a^2y^2-4a^3y+a^4=0,$$

revolves around the axis of y. Find the *campanulate* volume generated. If the same portion of the curve revolve around the axis of x, find the *fusiform* volume generated. Also, determine the area of this closed portion of the curve.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; W. C. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts; and the PROPOSER.

Solving the equation for x^2 we get $x^2=\pm\frac{1}{2}y\sqrt{y^2+8ay}-\frac{1}{2}(y^2+4ay-2a^2)$.

$\therefore$ The campanulate volume generated by the area MPA_1QNM is



$$V=\frac{\pi}{2}\int_0^a(y\sqrt{y^2+8ay}-y^2-4ay+2a^2)dy+$$

$$\frac{\pi}{2}\int_0^{\frac{1}{3}a}(y\sqrt{y^2+8ay}+y^2+4ay-2a^2)dy.$$

$$=\frac{\pi}{2}\Big[\tfrac{1}{3}(y^2+8ay)^{\frac{3}{2}}-2a(y-4a)\sqrt{y+8ay}+32a^3\log$$

$$\left\{y+4a+\sqrt{y^2+8ay}\right\}-\tfrac{1}{3}y^3-2ay^2+2a^2y\Big]_0^a+\frac{\pi}{2}\Big[\tfrac{1}{3}y^3+2ay^2-2a^2y+\tfrac{1}{3}(y^2+$$

$$8ay)^{\frac{3}{2}}-2a(y+4a)\sqrt{y^2+8ay}+32a^3\log\left\{y+4a+\sqrt{y^2+8ay}\right\}\Big]_0^{\frac{1}{3}a}$$

$$=\tfrac{4}{3}\pi a^3(12\log3-13).$$

[ZERR, MATZ, and BLACK.]

From the equation we get $y=\dfrac{2a(a^2-x^2)\pm(a^2-x^2)\sqrt{a^2-x^2}}{x^2+3a^2}$.

$\therefore$ The fusiform volume is

$$V=8a\pi\int_0^a\frac{(a^2-x^2)^2\sqrt{a^2-x^2}}{(x^2+3a^2)^2}dx=8\pi a^3\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{(4-\cos^2\theta)^2},\text{ where } x=a\sin\theta,$$

$$=8\pi a^3\left[\frac{17\theta}{2}-\frac{44\sqrt{3}}{3\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\tan\theta\right)+\frac{\cos\theta\sin\theta}{2}+\frac{8\sin\theta\cos\theta}{9\cos^2\theta+12\sin^2\theta}\right]_0^{\frac{1}{2}\pi}$$

$$=4\pi^2a^3\left(\frac{17}{2}-\frac{44\sqrt{3}}{9}\right)=\frac{2}{9}\pi^2a^3(153-88\sqrt{3}).$$

Also area is $A=2\displaystyle\int_0^a\frac{(a^2-x^2)\sqrt{a^2-x^2}}{x^2+3a^2}dx=2a^2\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{4-\cos^2\theta}=2a^2\left[\frac{8\sqrt{3}}{3}\right.$

$\left.\tan^{-1}\left(\dfrac{2}{\sqrt{3}}\tan\theta\right)-\dfrac{9\theta}{2}-\dfrac{\sin\theta\cos\theta}{2}\right]_0^{\frac{1}{2}\pi}=\dfrac{1}{6}\pi a^2(16\sqrt{3}-27).$

[ZERR, and MATZ.]

Or, fusiform volume$=2\pi\displaystyle\int_0^a(y_1^2-y_2^2)dx=16a\pi\int_0^a\frac{(a^2+x^2)^{\frac{5}{2}}dx}{(x^2+3a^2)^2}$

$$=16a^3\pi\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{(4-\cos^2\theta)^2}$$

$$=16a^3\pi\int_0^{\frac{1}{2}\pi}\left(\frac{\cos^6\theta}{4}+\frac{2\cos^8\theta}{4^3}+\frac{3\cos^{10}\theta}{4^4}+\frac{4\cos^{12}\theta}{4^5}+\ldots\ldots\right)d\theta$$

$$=8\pi^2a^3\left\{\frac{1.3.5}{2.4.6.4^2}+\frac{2.1.3.5.7}{2.4.6.8.4^3}+\frac{3.1.3.5.7.9}{2.4.6.8.10.4^4}+\ldots\ldots\right\},\text{ since}$$

$$\int_0^{\frac{1}{2}\pi}\cos^{4m}xdx=\frac{1.3.5\ldots\ldots(2m-1)}{2.4.6\ldots\ldots\ldots\ldots(2m)}\cdot\frac{\pi}{2}.$$

Area of closed portion$=2\displaystyle\int_0^a(y_1-y_2)dx+4\int_0^a\frac{(a^2-x^2)^{\frac{3}{2}}dx}{x^2+3a^2}$.

Let $x=a\sin\theta$, $A=4a^2\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{4-\cos^2\theta}=4a^2\int_0^{\frac{1}{2}\pi}\left(\frac{\cos^4\theta}{4}+\frac{\cos^6\theta}{4^2}+\frac{\cos^8\theta}{4^3}\right.$

$\left.+\dfrac{\cos^{10}\theta}{4^4}+\ldots\ldots\right)d\theta=2a^2\pi\left\{\dfrac{1.3}{2.4.4}+\dfrac{1.3.5}{2.4.6.4^2}+\dfrac{1.3.5.7}{2.4.6.8.4^3}+\ldots\ldots\right\}$,

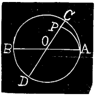
which series is also convergent. [BLACK.]

41. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A railroad turn-table 100 feet long is balanced upon a pivot in the center of a circular track 100 feet in diameter. How far does a man walk who starts at one end of the table and walks, at a uniform rate, the entire length of the table in the same time that the table makes two revolutions, if the table starts to turn at the same time the man starts to walk ?

Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

It is the purpose of this solution to find how far the man moves in space, if he always walks on the same line CD until across.



Let $OA=a$, $OP=r$, $\angle COA=\theta$, the velocity of C around the track n times the velocity of P along CD, P being the man's position at any time. Then $n.PC=\text{arc } AC$ $=a\theta$, $\therefore\ PC=\frac{a\theta}{n}$.

$\therefore\ r=a-PC=a-\frac{a\theta}{n}=\frac{a(n-\theta)}{n}$. $\therefore\ r=a(n-\theta)/n$

is the equation of the man's path; also, $ds=\sqrt{(dr)^2+r^2(d\theta)^2}$, but $(dr)^2=\frac{a^2}{n^2}(d\theta)^2$. $\therefore\ ds=\pm\frac{a}{n}\sqrt{1+(n-\theta)^2}d\theta$. $\therefore\ s=-\frac{2a}{n}\int\sqrt{1+(n-\theta)^2}d\theta$, for whole length $=\frac{a}{n}(n-\theta)\sqrt{1+(n-\theta)^2}+\frac{a}{n}\log\{n-\theta+\sqrt{1+(n-\theta)^2}\}+C$, but $2na=4\pi a$, $\therefore\ n=2\pi$, $\therefore\ r=a(2\pi-\theta)/2\pi$.

$s=\frac{a}{2\pi}(2n-\theta)\sqrt{1+(2\pi-\theta)^2}+\frac{a}{2\pi}\log\{2\pi-\theta+\sqrt{1+(2\pi-\theta)^2}\}$. The limits of θ are 0 and 2π, and $a=50$. $\therefore\ s=50\sqrt{1+4\pi_2}+\frac{25}{\pi}\log\left\{2\pi+\sqrt{1+4\pi^2}\right\}$.

$\therefore\ s=338.303$ feet.

Similarly solved by *G. B. M. ZERR*.

43. Proposed by J. C. NAGLE, M. A., C. E., Professor of Civil Engineering, A. and M. College, College Station, Texas.

Show that the volume included between the surface represented by the equation $z=e^{-(x^2+y^2)}$ and the xy plane equals the square of the area of the section made by the zx plane, the limits of x and y being plus and minus infinity.

I. Solution by Professor J. SCHEFFER, A. M., Hagerstown, Maryland.

Changing to polar co-ordinates, the volume is $=\int_0^{2\pi}d\theta\int_0^{\infty}e^{-r^2}rdr=$ $\frac{1}{2}\int_0^{2\pi}d\theta=\pi$. The area is $2\int_0^{\infty}e^{-x^2}dx$. Putting in the Gamma Function

$\int_0^\infty e^{-z}z^{n-1}dz=\Gamma(n)$, $z=x^2$, $n=\frac{1}{2}$, we find $2\int_0^\infty e^{-x^2}dx=\Gamma(\frac{1}{2})=\sqrt{\pi}$, which proves the assertion.

II. Solution by **G. B. M. ZERR, A. M., Ph. D.,** Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let V=the required volume ; A=the required area.

$$\therefore V=\iiint dxdydz=\iint zdxdy=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-(x^2+y^2)}dxdy.$$

$$\therefore V=\left[\int_{-\infty}^{+\infty}e^{-x^2}dx\right]\left[\int_{-\infty}^{+\infty}e^{-y^2}dy\right].\quad A=\iint dxdz=\int zdy=\int_{-\infty}^{+\infty}e^{-x^2}dx.$$

$$\text{But}\int_{-\infty}^{+\infty}e^{-x^2}dx=\int_{-\infty}^{+\infty}e^{-y^2}dy.\quad \therefore V=\left[\int_{-\infty}^{+\infty}e^{-x^2}dx\right]^2=A^2.$$

III. Solution by **C. W. M. BLACK, A. M.,** Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

$z=e^{-(x^2+y^2)}$. Applying formula for volume, $V=\iint zdydx$, we have $V=4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dydx$......(1). Also let $y=0$. Then $z=e^{-x^2}$ is the equation of section made by zx plane. Area$=2\int_0^\infty e^{-x^2}dx$........(2). Let this be equal to a......(3). Now put (1) in form of $V=4\int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}dydx$. Integrating with reference to y in accordance with (3), we have $V=2\int_0^\infty ae^{-x^2}dx=2a\int_0^\infty e^{-x^2}dx=a^2$, also in accordance with (3).

Professor William Hoover did not solve this problem but referred to Todhunter's Integral Calculus, Art. 204, where a good solution is given.

PROBLEMS.

49. Proposed by **B. F. BURLESON,** Oneida Castle, New York.

Find (1) in the leaf of the strophoid whose axis is a the axis of an inscribed leaf of the lemniscata, the node of the former coinciding with the crunode of the latter. Find (2) in a leaf of the lemniscata whose axis is b the axis a of an inscribed leaf of the strophoid, the node of the former also coinciding with the crunode of the latter.

50. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 329 East Second Street, N. Portland, Oregon.

A draw bridge, a feet in length, moves uniformly about a center axis. At the instant it began to open, a man stepped on the end; and, walking at a uniform rate in the straight line passing through its center, reached the opposite end just as it made n complete revolutions. Find the absolute path described by the man, and the ratio of his rate of motion in this path and the velocity of the end of the bridge. Apply the result to the case when $a=320$ and $n=2$.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

30. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

P is the lowest point on the rough circumference of a circle in a verticle plane at which a particle can rest, friction being equal to the pressure; to find the inclination of the radius through P to the horizon.

Solution by the PROPOSER.

If μ=the coefficient of friction, R=the normal reaction of the curve, μR= the friction, $=R$ by the problem. $\therefore\ \mu=1$.

W being the weight of the particle, we have, resolving along the tangent and radius through P,

$$W\sin\phi=\mu R\ldots\ldots(1).$$

$$W\cos\phi=R\ldots\ldots(2).$$

These give $\tan\phi=\mu=1$, or $\phi=\frac{\pi}{4}$.

Excellent solutions of this problem were received from *PROFESSORS ALFRED HUME, O. W. ANTHONY*, and *E. L. SHERWOOD*.

31. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A perfectly elastic, but perfectly rough mass M and radius R, rotating in a verticle plane with an angular velocity of ω, is let fall from a height, a, upon a perfectly elastic, but perfectly rough horizontal plane. Determine the motion of the body after striking the plane. What will be its ultimate motion?

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi, University P. O., Mississippi.

Let the angular velocity, ω, be in the direction of the motion of the hands of a clock.

Let ω' and v' be, respectively, the angular velocity, and the horizontal velocity of the center of the sphere, after the first impact.

The impulsive action at the point of contact is, then, Mv'.

The change in the angular momentum being equal to the moment of the impulse,

$$\tfrac{2}{5}MR^2(\omega-\omega')=Mv'R.$$

The surfaces being perfectly rough, there is no slipping and

$$v'=R\omega'.$$

$$\therefore\ \tfrac{2}{5}(\omega-\omega')=\omega',$$

$$\omega'=\tfrac{2}{7}\omega;$$

$$v'=\tfrac{2}{7}R\omega.$$

v'' and ω'' representing horizontal and angular velocities after second impact,

$$M(v''-v')=\text{impulsive friction},$$

$$\tfrac{2}{5}MR^2(\omega''-\omega')=-M(v''-v')R,$$

$$\tfrac{2}{5}(\omega''-\omega')=\omega'-\omega'',$$

$$\omega''=\omega'=\tfrac{2}{7}R\omega,\text{ and } v''=R\omega''=\tfrac{2}{7}R\omega.$$

The sphere moves on in an endless series of equal parabolas, with constant angular velocity and constant horizontal velocity, reaching the height a after every rebound.

Solutions of this problem were also received from Professors Zerr and Anthony. One or both of these solutions will appear in the next issue of the MONTHLY.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

30. Proposed by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

A and B are two integers, A consisting of 2 m figures each being 1, and B consisting of m figures each being 4. Prove that $A+B+1$ is a square.

I. Solution by H. W. DRAUGHON, Olio, Mississippi, and O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

Each of the integers is a Geometrical Series.

$A=1+10+100+$etc., to $2m$ terms, $=\frac{1}{9}(10^{2m}-1)$.

$B=4+40+400+$etc., to m terms, $=\frac{4}{9}(10^m-1)$.

$A+B+1=\frac{1}{9}(10^{2m}-1)+\frac{4}{9}(10^m-1)+1=\frac{1}{9}(10^{2m}+4.10^m+4)$
$=\{\frac{1}{3}(10+2)\}^2$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas, and H. C. WILKES, Skull Run, West Virginia.

$A=\frac{9}{16}B^2+\frac{1}{2}B$ as is shown by the following: Let $B-444$.

$\therefore \frac{9}{16}B^2+\frac{1}{2}B=111111$. This is true for any value of B.

Hence $A+B+1=\frac{9}{16}B^2+\frac{1}{2}B+1=\left(\frac{3B+4}{4}\right)^2=B^2$.

$\therefore A+B+1=(333......334)^2$, the number within the parenthesis consists of m figures. Let A_1 be an integer consisting of m figures all 1's.

Then $B^2=(333......334)^2=(B+1+A_1)^2$.

Also solved by *M. A. GRUBER* and *J. SCHEFFER*.

31. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

How many scalene triangles, of integral sides, can be formed with an altitude of 12? How many isosceles triangles?

I. Solution by ARTEMUS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington D. C.

1. To find right-angled triangles having one leg $=12$.

Let $x=$the required leg and $x+a=$the hypothenuse; then $(x+a)^2-x^2$ $=2ax+a^2=12^2=144$; whence $x=\dfrac{144-a^2}{2a}$.

It is easily seen that a must be even, and that it cannot exceed 10; but as x must be integral a can only be 2, 4, 6, or 8.

Take $a=2$, then $x=35$; take $a=4$, then $x=16$; take $a=6$, then $x=9$; take $a=8$, then $x=5$. Hence there are four right-angled triangles having one leg $=12$, viz: 12, 35, 37; 12, 16, 20; 12, 9, 15; 12, 5, 13.

2. Any two right-angled triangles, p, c, a; p, b, d, can be combined in two different ways to form a scalene triangle, giving the triangles a, b, $c+d$; a, b, $c-d$. Hence the four right-angled triangles found above can be combined two and two in two different ways to form scalene triangles; therefore there are twelve such triangles which have an altitude of 12, as follows: 13, 14, 15; 20, 37, 51; 15, 20, 25; 15, 37, 44; 13, 37, 40; 13, 20, 21; 13, 15, 4; 20, 37, 19; 15, 20, 7; 15, 37, 26; 13, 37, 30; 13, 20, 11.

There can be only four isosceles triangles with integral sides having an altitude of 12, viz: 13, 13, 10; 15, 15, 18; 20, 20, 32; 37, 37, 70.

II. Solution by A. H. BELL, Box 184, Hillsboro, Illinois.

We evidently require to find two □ numbers whose difference shall be equal to any given number. Let $x=$the side of the lesser square, and $d=$to

two unequal factors $=ab$, $a>b$; let $x+b=$the greater square.

Then $(x+b)^2-x^2=ab$, and $x=\frac{a-b}{2}$, $x+b=\frac{a+b}{2}$.

The unequal factors of the difference $(12)^2$ are 2×72, 4×36, 6×24, 8×18; these give for sides of squares in the formula, and complete the following right-angled triangles, in the order of altitude, base and hypotenuse: 12, 5, 13; 12, 9, 15; 12, 16, 20; 12, 35, 37.

By doubling the base of each will give four isosceles, and by adding and subtracting the bases from each pair will give 12 scalene triangles.

III. Solution by the PROPOSER.

All scalene $\triangle$'s are rt. $\triangle$'s or are the sum or the difference of two rt. $\triangle$'s of equal altitudes. The $\triangle$'s of this problem are restricted to $\triangle$'s of integral sides having an altitude of 12.

We first find the rt. $\triangle$'s of integral sides having an altitude of 12. These are four in number: 12, 5, 13; 12, 35, 37; 12, 9, 15; and 12, 16, 20.

Then, by *sum* and *difference*, we form combinations by twos by joining their equal altitudes. It will readily be seen, if $n=$the number of rt. $\triangle$'s of a given altitude, that the number of combinations each by *sum* and by *difference* of twos is the sum of the series, $n-1$, $n-2$, $n-3$......1. The sum of this series is $\frac{n(n-1)}{2}$. As there are two such series, the number of combinations is $n(n-1)$.

Adding to this the n rt. $\triangle$'s, we find the total number of scalene $\triangle$'s to be n^2, which is the square of the number of rt. $\triangle$'s having the given altitude. Hence the number of scalene $\triangle$'s of integral sides having an altitude of 12 is $4^2=16$.

All isosceles $\triangle$'s of integral sides are the union of two equal rt. $\triangle$'s by joining the altitudes. There are as many isosceles $\triangle$'s of integral sides having a given altitude as there are rt. $\triangle$'s of integral sides having the given altitude. Hence there are four isosceles $\triangle$'s of integral sides having an altitude of 12.

Also solved by *O. W. ANTHONY*, *H. W. DRAUGHON*, *G. B. M. ZERR*, and *WILLIAM HOOVER*.

32. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Decompose into its prime factors the number 549755813889.

Solution by the PROPOSER.

To find the factors of $2^{39}+1=549755813889$. The old masters have demonstrated that prime factors of a^n+1 must be of the general form of $2nx+1$. Suppose we take $a^{mn}+1$, mn odd, the factors of mn are m, n, 1; then the prime divisors will be of form $a^{mn}+1$, a^n+1, and $a+1$. Divide out these factors; the $\sqrt{\text{balance}}$ will show the limit of the trial divisors which must be of the general form $2mnx+1=$to prime form of factors$=8mnx+1$ and $8mnx+(6mn+1)$, if these will not or if they do divide the balance, we conclude the balance to be a prime number.

Solution of $2^{39}+1=549,755,813,889\div$by divisors (prime) $2^{13}+1$, 2^3+1,

and $2+1$; then $3^2.2731.(22366891).\sqrt{22366891}=4620$ limit of divisors of the form prime $8mnx+1$ and $8mnx+(6mn+1)=312x+1$ and $312x+235$, and they are 313, 547, 859, 937, 1171, 1249, 1483, 1873(2731)3121, 3433, 4057, 4603 to limit, none of which will divide the balance, hence 22366891 is prime.

$\therefore$ factors are $3^2\times2731\times22366891$.

No solution of Problem 33 has been received.

PROBLEMS.

43. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the series of integral numbers in which the sum of any two consecutive terms is the square of their difference.

44. Proposed by A. H. HOLMES, Box 963. Brunswick, Maine.

The hypotenuse of a right-angled triangle ABC, right-angled at A, is extended equally at both extremities so that $BE=CD$. Draw AD and AE. Find integral values for all the lines in the figure thus made.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

NOTE ON AVERAGE AND PROBABILITY WITH REFERENCE TO THE SOLUTIONS OF PROBLEM 26, pp, 282-83, AND 327-28.

By ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington D. C.

I WILL remark at the outset that, unfortunately, mathematicians are not agreed as to the method of solving certain problems in Average and Probability. The difference of opinion in some cases relates to the interpretation of the meaning of the problem, and in others to the quantity that should be considered as the independent variable, and between what limits taken, and again as to whether the "points" are uniformly distributed along a certain line or over a certain surface, etc.

If points be uniformly distributed on a line, the *number* of points is proportional to the length of the line; and if points be uniformly distributed over a surface, the number of points is *proportional* to the area of the surface, etc.; but if the points be *not* uniformly distributed, then the line or surface can not be taken as a *true* measure of the number of points.

Problem 26. "Find the average of all right-angled triangles having a given hypotenuse."

Prof. Matz' first solution, p. 82, would be correct if he had taken the limits of x from 0 to h instead of from 0 to $\frac{1}{2}h\sqrt{2}$. He supposes one of the legs to increase uniformly from 0 to $\frac{1}{2}h\sqrt{2}$, or till the legs become equal; but this assumption does not give *all* the triangles because, while x *increases* uniformly from 0 to $\frac{1}{2}h\sqrt{2}$, $\sqrt{(h^2-x^2)}$ does not decrease uniformly from h to $\frac{1}{2}h\sqrt{2}$. The limits should be 0 and h, for if one leg varies uniformly from 0 to h *all possible* right-angled triangles will be generated, and the number of the triangles will be proportional to h.

If he had taken x from 0 to h in his first solution, and θ from 0 to $\frac{1}{2}\pi$ instead of from 0 to $\frac{1}{4}\pi$ in the second, he would have obtained in both solutions the result $\frac{1}{6}a^2$, which I believe to be correct.

In the second method of solution, adopted by Prof. Zerr and others, and approved by the Editor, it is assumed that one of the *acute angles* varies uniformly, and that the *number* of triangles is proportional to the semicircumference whose diameter is the given hypotenuse.

The vertices of the right angles of all possible right-angled triangles having a given hypotenuse a will be posited on a semicircumference whose diameter is a, but *will not* be uniformly distributed thereon; hence the semicircumference *can not* be taken as the true measure of the number of triangles.

I most emphatically dissent from the conclusion announced by the Editor in the last line of p. 328. The last paragraph of the note is sound down to the last line, but it does not by any means necessarily follow from any statement made therein that "the solutions leading to the result $\dfrac{a^2}{2\pi}$ are the *correct* and *only* solutions of the problem." I hold that the solution given by Prof. Anthony on p. 283, and previously given by myself in the *Mathematical Magazine*, leading to the result $\frac{1}{6}a^2$ (misprinted $\frac{1}{3}a^2$ in the first line of the Editor's note), is the *true* solution of the problem.

The conception of a triangle is from its sides; and if we cause one of the legs to take all possible values from 0 to a it is very clear to me (and ought to be to every one) that all possible right-angled triangles having that hypotenuse will be formed.

The problem as proposed is *definite* as the Editor correctly states in his note, and requires the "average area of *all* the right-angled triangles having a given hypotenuse"; but the solution which he asserts the "*correct* and *only*" one restricts the triangles to those having the vertices of their right angles uniformly distributed on the semicircumference whose diameter is the given hypotenuse, and therefore is not a solution of the problem proposed, but of the following problem, viz: Required the average area of the right-angled triangles having a given hypotenuse and the vertices of their right angles uniformly distributed on the semicircumference whose diameter is the given hypotenuse.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

Why was the letter π chosen to represent $\frac{c}{d}$?

LOTTIE SMITH, Houston, Miss.

I have several problems, which though solved by quadratics, have one positive root and no other. I have also several problems, which can only be solved by quadratics, which have two roots, one real and one imaginary, notwithstanding that "imaginary expressions enter an equation by pairs," which at present I will not disclose. The following problem is from Bell's Algebra (Chamber's Edition Course): "Given $\sqrt{(2x^2-2)}=3x-5$. $x=3$, or $\frac{9}{7}$." The $\frac{9}{7}$ fails to verify the equation. Can another root besides 3 be found that will?

R. GREENWOOD, Morris, Ill.

COMMENTS ON PROBLEM 11—GEOMETRY.

What does the gentleman do with the parts of the circle outside of his own central circle and the seven circles he gives to his seven children? If he does not "retain" it, he must think that these pieces will suit his wife.

W. F. BRADBURY,
Cambridge Latin School, Cambridgeport, Mass.

When the condition of the problem is satisfied, one of the seven equal circular farms will be concentric with the original farm. This condition is, therefore, incompatible with the (insinuated) condition that the gentleman shall retain for himself an area about the center of the original farm. The "problem" is merely a puzzle.

L. E. PRATT, Tecumseh, Neb.

On page 249 of Wentworth's *College Algebra*, we find the author conclude that $\underline{|0}=1$, *i. e.*, factorial zero is equal to unity. On page 246 the definition of *factorial* is given: $\underline{|n}=n(n-1)(n-2)\ldots\ldots1$, *i. e.*, factorial n is equal to the product of all the natural numbers from n to 1 inclusive. If n were 3, we would have $\underline{|3}=3.2.1$; if $n=8$, then $\underline{|8}=8.7.6\ldots\ldots1$. So for any other number. If therefore 0 is to be one of them, it must submit to the same law.

$\therefore\ \underline{|0}=0\ldots\ldots1=0.1=0$! This would show that factorial zero, if it has any meaning at all, must be equal to zero. But *factorial zero* is not comprised within the definition given by the author: by that definition the first factor of the product is the number given (n), the last is unity, (1), which therefore excludes 0. The mistake made by the author in arriving at the result consists in disregarding the factor 0 in one of the terms of a fraction. From the formula

$C_{n,r} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}}$ is derived, by multiplying each term by $\underline{|n-r}$, the formula: $C_{n,r} = \frac{n(n-1)\ldots\ldots(n-r+1)(n-r)\ldots\ldots1}{\underline{|r}\times(n-r)\ldots\ldots1}\ldots[A]$,

or $C_{n,r} = \frac{\underline{|n}}{\underline{|r}\ \underline{|n-r}}$. Now if in this latter we make $n=r=1$ we obtain

$$C_{1,1} = \frac{\underline{|1}}{\underline{|1}\ \underline{|1-1}} = \frac{1}{1.\underline{|0}} = \frac{1}{\underline{|0}}\ldots\ldots(B).$$

[But as the number of combinations made of one element with one in the group is one, we also have

$C_{1,1}=1. \quad \therefore 1=\frac{1}{\underline{|0}} \quad \therefore \underline{|0}=1.$]

But suppose we make the substitution in formula [A];

$$C_{n,r} = \frac{n(n-1)\ldots\ldots(n-r+1)(n-r)\ldots\ldots1}{\underline{|r}\,(n-r)\ldots\ldots1}$$

where $(n-r+1)=(1-1+1)=1$.

$$C_{1,1} = \frac{(1)(n-r)\ldots\ldots1}{\underline{|1}\,(n-r)\ldots.1} = \frac{1}{1}\cdot\frac{\underline{|n-r}}{\underline{|n-r}} = \frac{1}{1} = 1 \text{ and not: } \frac{1}{\underline{|0}}.$$

The factor which becomes 0 in the denominator also occurs in the numerator and is to be cancelled. The error made as in [B], consists in neglecting this factor in the numerator but retaining it in the denominator.

OSCAR SCHMIEDEL,
Bethany College, Bethany, West Virginia.

ANSWERS TO QUERIES IN MONTHLY FOR MARCH, 1894, (VOL. 1, NO. 3, P. 103.)

BY PROF. JOHN N. LYLE, FULTON, MO.

I. Whether Lobatschewsky's theorem 4 is "sound" or not depends upon what shall be regarded as "sound" in geometry. If the assumption that a plane is the surface of a sphere and that two straight lines drawn therein perpendicular to a third do intersect is sound; then Lobatschewsky's theorem 4, since it contradicts this assumption, must be unsound. Otherwise, two propositions that contradict each other may both of them be sound. Again, if the soundness of Euclid's propositions 27 and 28, Book 1. is granted, that of Lobatschewsky's theorem 4 must also be conceded, since is a legitimate corollary of those propositions.

II. Lobatschewsky's theorem 4 which reads as follows: "Two straight

lines perpendicular to a third never intersect, how far soever they be produced" contradicts flatly the assumption that these perpendiculars do intersect, no matter where the intersection is supposed to occur. The *fact* and not the *place* of supposed intersection constitutes the contradiction. Von Staudt's assumption that two straight lines perpendicular to a third have "at infinity a common point" contradicts proposition 27, Book 1. of Euclid's Elements, and hence can not be in harmony with it. Euclidean space cannot be extended to any point of intersection of the two perpendiculars under notice for the good and sufficient reason that those perpendiculars do not and can not intersect in that space.

III. No. IV. Yes.

V. No, for the reason that it involves contradiction. By definition every straight line having two ends is finite. Hence, to affirm that such a line is infinite in length is to attribute to it contradictory attributes. No infinite straight line can be drawn between two points located in space and geometrical science does not concern itself with what is supposed to occur or not to occur outside of space. Juggling with algebraical symbols can not alter the cold, hard facts of the Euclidean geometry.

VI. In his theorem 16 Lobatschewsky is *studiously silent* as to whether he regards the boundary line itself as a *cutting* or a *not-cutting* line. In his theorem 33, however, he uses this language—"hence not only does the distance between two parallels decrease (Theorem 24), but with the prolongation of the parallels towards the side of the parallelism this at last wholly vanishes. Parallel lines have therefore the character of asymptotes." From this it appears that Lobatschewsky holds that the distance between asymptotes and their curves "at last wholly vanishes."

VII. In theorems 32 and 33 Lobatschewsky exhibits without disguise his use and interpretation of the symbols 0 and ∞, and his speculative opinions respecting geometrical data that dominate his thinking and thus determine his conclusions. The reason assigned by Lobatschewsky for his conclusion that the distance between parallels decreases and "at last wholly vanishes" is that $s'=0$ for $x=\infty$ in the formula $s'=se^{-x}$. There is nothing novel, brilliant or profound in manipulating algebraical symbols in such fashion. It is in fact a familiar game of analytical sophistry more than two hundred years old played in the school of Leibnitz with 0 and ∞ as dice. In his theorem 32 Lobatschewsky informs us that "A circle with continually increasing radius merges in the boundary line." He further says that "one may also call the boundary line *a circle with infinitely great radius.*"

When Lobatschewsky rejected Euclid's axiom 12 and accepted in its stead a straight line as the circumference he evidently strained at a gnat and swallowed a camel. In Lotze's Metaphysics, Part II., Vol. I., pages 290 and 291 the fol-

lowing extract is found : "A finite arc of a circle, of course, becomes perpetually more like a straight line as the radius of the circle to which it belongs is increased; but the whole circle never comes to be like one. However infinitely great we may conceive the radius as being, nothing can prevent us from conceiving it to complete its rotation around the center, and till such rotation is completed we have no right to apply the conception of a circle to the figure which is generated: discourse about a straight line which being in secret a circle of infinite diameter, returned into itself, is not a portion of esoteric science, but a proof of logical barbarism. Just the same is shown by phrases about parallel lines which are supposed to cut each other at an infinite distance. They do not cut each other at any finite distance, and as every distance when conceived as attained would become finite again, there is simply no distance at which they do so ; it is utterly inadmissible to pervert this negation into the positive assertion that in infinite distance there is a point at which intersection occurs."

EDITORIALS.

We were compelled to omit the Department of Geometry in this issue because of lack of sorts, and the Miscellaneous Department because this number has now grown far beyond its proper limits.

No pains will be spared on the part of the editors to make Vol. III. of great value to all its readers. To this end, we trust that we may have the coöperation of all of our old contributors and that of many new ones.

Professor E. L. Sherwood should have been given credit for solving Problem 46, Department of Geometry. Editor Colaw and Prof. Cooper D. Schmitt each sent a solution of Problem 54, Department of Arithmetic, but too late for credit in the proper place.

A correspondent who has a large collection of mathematical autographs and MSS. will exchange duplicates with any other who is interested in the same line. Professor Finkel will put this correspondent in communication with any one who will send his address.

In order that we may increase the subscription list of the MONTHLY, we invite each of our old subscribers to take advantage of the following offer :

To any old subscriber sending us the names of three new subscribers, and six dollars, we will send THE AMERICAN MATHEMATICAL MONTHLY one year as a premium. This offer ought to quadruple the number of our subscribers.

While much is being said in the literary world about endowing magazines, what is wrong with making the MONTHLY an example of endowed periodicals?

This year a great friend of the MONTHLY and a Professor of Mathematics in an eastern college, invested $60. in extra copies. If one hundred of our subscribers would donate $50. towards an endowment fund, they would be entitled to a perpetual subscription and the MONTHLY saved from the fate of its predecessors,—discontinuance in the course of a decade or two.

This number completes the second volume of the MONTHLY, and though its success in the two years during which it has been issued has not been what we hoped for, it has not been altogether discouraging. We are encouraged by words from various mathematicians of great eminence that the MONTHLY is growing in influence and favor. We believe this to be true. A glance at our list of contributors will show that it includes the best mathematicians in America. Having the support of the ablest mathematicians of this country, the MONTHLY should continue to appear each month during the year that is now upon us. The editors have, therefore, no thought of discontinuing its publication, and we trust that we may have the earnest support of all of our old subscribers in the still further enhancing of its worth. In the January number we shall use a better quality of paper and thus improve its appearance. We have on hand a number of very excellent articles from leading mathematicians which will appear during the coming year. Dr. Halsted will continue his translation of Saccheri's geometry, and Dr. Miller will continue his articles on Substitution Groups. Dr. Moore has furnished an article on An Interesting System of Quadratic Equations, which will appear in the January number. Prof. Zerr has furnished an article on the Centroid of Plane Areas, the first part of which will also appear in the January number. A great many other papers of interest and importance from prominent mathematicians may be expected. The January number will contain an interesting biography of the great Russian Mathematician, Wolfgang Bolyai, by Dr. Halsted. Other biographies of noted mathematicians will be published during the year.

A great many of our subscribers are in arrears on subscription for Vol. I. and Vol. II. We shall be greatly obliged if those owing us will kindly remit at once, as we are much in need of funds. Please send money by Draft or Post-office Money Order to B. F. Finkel, 1320 Washington Avenue, Springfield, Mo.

BOOKS AND PERIODICALS.

High School Mathematical Teaching and Text-Books. A monograph from the *Inland Educator.* By Robert J. Aley, A. M., Professor of Mathematics in the University of Indiana, Bloomington, Indiana.

In this little pamphlet of 20 pages, Professor Aley has given some good hints on the teaching of Mathematics in the High School. He blames the teacher, the text-book, or

both for the hatred so many boys and girls have for Mathematics. He says, page 13, "This feeling is not natural to the normally constituted mind. Psychologists and educators generally tell us that the mind ought to find pleasure in mathematical exercises. There is no subject in which the student can so early begin making discoveries for himself. The mere beginner in geometry can make and solve exercises that would have made Appollonius famous. This hatred for mathematics must then in general be the fault of the teacher, or of the text-book, or of both. Whichever it may be, it is in the province of the teacher to remove."

Wilkes' Rules of Multiplication. By H. C. Wilkes, Skull Run, West Virginia.

In this little pamphlet of 16 pages, Mr. Wilkes has given a number of rules for the rapid multiplication of two numbers. He says, page 1, "To be expert in multiplying, three things are essential: 1st, To be able to multiply any number by a single digit, operating from left to right; 2nd, To know instantly from memory the product of any two numbers each less than 20; 3rd, To be able to add mentally and quickly any two numbers each less than 100." We give a single Rule: *How to mentally see the product of any two numbers in the "teens."* Example: 19×16

$$\begin{array}{r} 19 \\ 6 \\ 54 \\ \hline 304 \end{array}$$

Place the unit figure of one of the numbers under the other number, and then place the product of the unit figures as shown, and add all together.

Laboratory Methods of Teaching Mathematics in Secondary Schools. By Adelia R. Hornbrook, A. M., Teacher of Mathematics in High School, Evansville, Indiana, and author of Concrete Geometry for Common and Grammar Schools. Pamphlet, 16 pages. Chicago: American Book Co.

In this little pamphlet, Mrs. Hornbrook has given some timely recommendations on the Laboratory Method of Teaching Mathematics from a psychological standpoint. She proves conclusively that the Laboratory Method which has worked wonders in the Department of Natural Science, can also be made to produce equally good results in Mathematics if sufficient care is taken on the part of the teacher. The term "Laboratory Method" as applied to teaching Mathematics means the method of independent personal investigation on the part of the learner under the leadership of a teacher who furnishes only the necessary aids to interpretation. The pamphlet is well worth a careful reading.

Plane and Solid Geometry. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in the Michigan State Normal School. 8vo. cloth and leather back, 320 pp. Boston: Ginn & Co.

In the last issue of the MONTHLY, we announced that Drs. Beman and Smith had written a Geometry and that something new along the line of Geometry might be expected. The book is now ready and we are quite sure that it will meet with public favor. It has a number of very strong points in its favor. We can only mention a few of these. (1) It invests the geometry of the Ancients with something of the spirit of Modern Mathematics; (2) Many terms that are not new but are rarely found in similar works have been freely used and thus the student is made familiar with a nomenclature that is very essential to the study of Modern Higher Mathematics; (3) Methods of attack are suggested early, and at

the end of Book III. are treated with considerable fullness; (4) Historical notes, which have a tendency to relieve the monotony of class routine and awaken interest in even the most stupid and indifferent student, are frequently inserted; (5) A biographical table containing the names of forty-two mathematicians, Ancient and Modern, who have been instrumental in shaping the course of Mathematics, is appended; and (6), a table of Etymologies is also appended.

The value of a table of Etymologies can not be over estimated. The student coming in contact with new terms will naturally be interested in the etymology, a study of which will fix the meaning in his mind. I believe that it would have proved even more valuable had the etymology been given in connection with the term when it is used in the text.

The principles of Duality and Continuity are illustrated and explained. The book contains 783 exercises for original work. The book is a most admirable one and we take pleasure in recommending it to any who are seeking a good text on geometry.

B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single number, 10 cents. Irvington-on-the-Hudson, New York.

No one ever thought of introducing so expensive a feature as lithographic color work in the days when the leading magazines sold for $4.00 a year and 35 cents a copy. But times change, and the magazines change with them. It has remained for *The Cosmopolitan*, sold at one dollar a year, to put in an extensive lithographic plant capable of printing 320,000 pages per day (one color). The January issue presents as a frontispiece a water-color drawing by Eric Pape, illustrating the last story by Robert Louis Stevenson, which has probably not been excelled even in the pages of the finest dollar French periodicals. The cover of *The Cosmopolitan* is also changed, a drawing of page length by the famous Paris artist Rossi, in lithographic colors on white paper takes the place of the manilla back with its red stripe. Hereafter the cover is to be a fresh surprise each month.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York.

During the closing weeks of 1895 the daily papers have published an extraordinary amount of interesting and important news. It is worth something to the busy newspaper reader to have this mass of information taken up, arranged, digested and reviewed in a calm and intelligent manner. The *Review of Reviews* performs this service very efficiently every month. The number for January, 1896, is especially strong in this respect. The editorial department called "The Progress of the World," is distinguished for its able handling of national and international topics of the hour. In fact, the *Review* occupies a unique position as a truly "international magazine." Its soundly "American" stand on the Venezuelan question is significant.

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SOLUTIONS AND PROBLEMS.

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